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Unsteady conjugated heat transfer in laminar pipe flow

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Abstract—This study analyses the transient conjugated heat transfer in laminar pipe flow, where the flow is both hydrodynamically and thermally fully developed. Two cases are considered: a prescribed constant wall temperature and a constant heat flux at the wall. A non-standard method of separation of variables is applied, which treats the fluid and the solid as one region with certain discontinuities. The resulting eigenfunctions, which are not orthogonal to each other with respect to the usual weight function according to the Sturm–Liouville theorem, are made orthogonal to each other with respect to a special weight function. It is concluded that the degree of conjugation and viscous dissipation may have a great impact on the temperature distribution in the fluid.

INTRODUCTION

WE CONSIDER the transient conjugated heat transfer in laminar pipe flow. The flow is hydrodynamically and thermally fully developed and either the wall temperature or the wall heat flux is prescribed.

The need to solve the coupled temperature distribution in the fluid and the solid arises in the design of energy-related systems during the period of startup, shutdown or any off-normal surge in normal operating conditions. Examples are the design of compact heat exchangers, thick-walled pipes, change in power level transients in gas turbine engines and the design of cooling channels of nuclear reactors. In such cases, the boundary conditions imposed at the external surface, in general, are different from their counterparts at the internal surface. These deviations emerge because the wall plays a significant role in distributing the heat transferred from the external surface to the fluid or vice versa.

Early attempts to solve unsteady heat transfer problems mainly employed approximate methods to deduce the gross features of the problems [1–3]. Recently, several numerical solutions have been obtained for unsteady thermal entrance heat transfer in laminar channel flows with various thermal boundary conditions [4–6]. All the noted studies [1–6] focused on the transient heat transfer characteristics, neglecting the heat conduction in the solid and its heat capacity. The results of such investigations are valid for flows in thin-walled ducts, but not for thick-walled ones. Recently, Sucec and Sawant [7–10] and Cotta *et al.* [11] analysed the effect of wall heat capacity on unsteady heat transfer in laminar channel flows and showed that it is of great importance. Nevertheless, wall conduction still remained untreated. The latter effect was examined by Chung and Kassemi [12], Krishan [13], Lin and Kuo [14] and Yan *et al.* [15]. The methods of solutions were either approximate [12, 13] or numerical [14, 15]. We address the problem studied by Krishan [13] who employed the method of Laplace transforms, but the resulting solutions were valid for short-time periods only. In the following the problem is first formulated. An exact solution is derived by an extended method of separation of variables developed by Yeh [16, 17], which is capable of treating discontinuous eigenfunctions. Methods for the solution of multispace domain diffusion problems can be found in an excellent study by Wirth and Rodin [18], who mentioned over 200 references. Detailed results are presented and compared to those of Krishan [13].

ANALYSIS

Consider a hydrodynamically and thermally fully-developed flow with mean velocity u_m in an infinitely long pipe. The internal radius of the pipe is R_i and its external radius is R_o. The solution is carried out for the first time domain $t < t_{max} = z/2u_m$, where z is an axial distance from the inlet cross-section. In this domain the downstream locations in the pipe have not yet been reached by any of the fluid that was at the pipe inlet at the beginning of the transient. For the boundary conditions considered, $\partial T/\partial z = 0$, so that the energy equations for heat transfer in the fluid and solid, respectively, are given by

NOMENCLATURE

A_n constant, equation (30)
$\mathbf{A}$ matrix, equation (28)
B_1 constant, equation (34)
B_2 constant, equation (35)
B_3 constant, equation (34)
B_4 constant, equation (35)
B_n constant, equation (30)
b ratio of external and internal radii of the pipe, R_o/R_i, or variable, equation (35)
c dimensionless magnitude of viscous dissipation
c_n constant, equation (50) or equation (61)
c_p specific heat at constant pressure
D_1, D_2 constants, equation (32)
E_1, E_2 constants, equation (33)
e_1, e_2 constants, equation (26)
F constant, equation (41)
f function, see definition below equation (15)
f_1, f_2 constants, equation (26)
G_n constant, equation (45)
G_n^* constant, equation (60)
g function, see definition below equation (15)
g^* function, equation (58)
K ratio of conductivities, $\bar{k}_2/\bar{k}_1$
K^* conjugation parameter, $\sqrt{k}/K$
k ratio of wall to fluid thermal diffusivities, $\rho_1 c_{p1}\bar{k}_2/\rho_2 c_{p2}\bar{k}_1$
$\bar{k}$ thermal conductivity
N number of terms in series expansion
p function, equation (31)
q heat flux or function, equation (31)
R space eigenfunction, equation (26)
R_i internal pipe radius
R_o external pipe radius
r dimensionless radial coordinate, $\bar{r}/R_i$
$\bar{r}$ radial coordinate
S_1 dimensionless heat flux
S_2 dimensionless wall temperature
T temperature
T_i initial system temperature
T_w wall temperature
T_R reference temperature
t time
u_m mean velocity of fluid
V velocity distribution
w weight function, equation (31)
x independent variable
$\mathbf{x}$ vector, equation (37)
y dependent variable, equation (31).

Greek symbols
Γ time function related to the separation of variables, equation (48)
θ dimensionless temperature distribution
λ separation constant
μ dynamic viscosity
ρ density
τ dimensionless time, $\bar{k}_1 t/\rho_1 c_{p1} R_i^2$
ϕ function, equation (56)
ψ dimensionless temperature distribution, equation (51).

Superscripts
$-$ fluid region
$+$ solid region.

$$\rho_1 c_{p1}\frac{\partial T_1}{\partial t} = \bar{k}_1 \frac{1}{r}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial T_1}{\partial \bar{r}}\right) + \mu\left(\frac{\partial v}{\partial \bar{r}}\right)^2,$$
$$0 < t < t_{\max}, \quad 0 < \bar{r} < R_i \tag{1}$$

$$\rho_2 c_{p2}\frac{\partial T_2}{\partial t} = \bar{k}_2 \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial T_2}{\partial \bar{r}}\right),$$
$$0 < t < t_{\max}, \quad R_i < \bar{r} < R_o \tag{2}$$

and the associated initial and boundary conditions are

$$t = 0, 0 \leqslant \bar{r} \leqslant R_o: \quad T_1 = T_2 = T_i \tag{3}$$
$$t > 0, \bar{r} = 0: \quad T_1 = \text{finite (or } \partial T_1/\partial \bar{r} = 0) \tag{4}$$
$$t > 0, \bar{r} = R_i: \quad T_1 = T_2 \tag{5}$$
$$t > 0, \bar{r} = R_i: \quad \bar{k}_1 \partial T_1/\partial \bar{r} = \bar{k}_2 \partial T_2/\partial \bar{r} \tag{6}$$
$$t > 0, \bar{r} = R_o: \quad -\bar{k}_2 \partial T_2/\partial \bar{r} = q \quad \text{or} \quad T_2 = T_w \tag{7a,b}$$

where subscripts 1 and 2 refer to the fluid and the solid, respectively. All variables retain their usual meaning and are given in the Nomenclature.

Define the following dimensionless variables:

$$K = \frac{\bar{k}_2}{\bar{k}_1}, \quad r = \frac{\bar{r}}{R_i}, \quad \tau = \frac{\bar{k}_1 t}{\rho_1 c_{p1} R_i^2}, \quad k = \frac{\bar{k}_2 \rho_1 c_{p1}}{\bar{k}_1 \rho_2 c_{p2}}$$

where for an external wall heat flux boundary condition

$$\theta = \frac{k_1(T - T_i)}{qR_i}, \quad S_1 = -\frac{\bar{k}_1}{\bar{k}_2}, \quad c = \frac{16\mu u_m^2}{qR_i}$$

while for a wall temperature boundary condition

$$\theta = \frac{T - T_i}{T_w - T_R}, \quad S_2 = \frac{T_w - T_i}{T_w - T_R}, \quad c = \frac{16\mu u_m^2}{\bar{k}_1(T_w - T_R)}$$

with $v = 2u_m(1 - r^2/R_i^2)$ to obtain the following mathematical formulation of the problem:

$$\frac{\partial \theta_1}{\partial \tau} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta_1}{\partial r}\right) + cr^2, \quad \tau > 0, 0 < r < 1 \quad (8)$$

$$\frac{\partial \theta_2}{\partial \tau} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta_2}{\partial r}\right), \quad \tau > 0, 1 < r < b \quad (9)$$

$$\tau = 0, 0 \leqslant r \leqslant b: \quad \theta_1 = \theta_2 = 0 \quad (10)$$

$$t > 0, r = 0: \quad \theta_1 = \text{finite (or } \partial\theta_1/\partial r = 0) \quad (11)$$

$$\tau > 0, r = 1: \quad \theta_1 = \theta_2 \quad (12)$$

$$\tau > 0, r = 1: \quad \partial\theta_1/\partial r = K\partial\theta_2/\partial r \quad (13)$$

$$\tau > 0, r = b: \quad \partial\theta_2/\partial r = S_1 \quad \text{or} \quad \theta_2 = S_2. \quad (14a,b)$$

Equations (8)–(14) can be put in the following unified form:

$$\frac{\partial \theta}{\partial \tau} = f(r)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) + g(r), \quad \tau > 0, 0 < r < b \quad (15)$$

with

$$f(r) = \begin{cases} 1 \\ k \end{cases}, \quad g(r) = \begin{cases} cr^2 & 0 \leqslant r \leqslant 1 \\ 0 & 1 \leqslant r \leqslant b \end{cases}$$

$$\tau = 0, 0 \leqslant r \leqslant b: \quad \theta = 0 \quad (16)$$

$$\tau > 0, r = 0: \quad \theta = \text{finite} \quad \text{or} \quad \partial\theta/\partial r = 0 \quad (17)$$

$$\tau > 0, r = 1: \quad \theta^- = \theta^+ \quad (18)$$

$$\tau > 0, r = 1: \quad \partial\theta^-/\partial r = K\partial\theta^+/\partial r \quad (19)$$

$$\tau > 0, r = b: \quad \partial\theta/\partial r = S_1 \quad \text{or} \quad \theta = S_2 \quad (20a,b)$$

where θ^- and θ^+ are the temperature distributions in the liquid and solid, respectively.

Thus, the fluid and solid domains are converted into a single region with a discontinuous heat source and thermophysical properties.

The eigenvalue problem associated with equation (15) can be put in the form

$$f(r)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = -\lambda^2 R \quad (21)$$

$$r = 0: \quad R = \text{finite (or } \partial R/\partial r = 0) \quad (22)$$

$$r = 1: \quad R^- = R^+ \quad (23)$$

$$r = 1: \quad \partial R^-/\partial r = K\partial R^+/\partial r \quad (24)$$

$$r = b: \quad \partial R/\partial r = 0 \quad \text{or} \quad R = 0 \quad (25a,b)$$

where R is an eigenfunction and $-\lambda^2$ is a separation constant. The solution of equation (21) can be expressed by

$$R = \begin{cases} R_1 = e_1 J_0(\lambda r) + e_2 Y_0(\lambda r) & 0 \leqslant r \leqslant 1 \\ R_2 = f_1 J_0(\lambda r/\sqrt{k}) + f_2 Y_0(\lambda r/\sqrt{k}) & 1 \leqslant r \leqslant b. \end{cases} \quad (26)$$

To satisfy condition (22), e_2 must be zero and the remaining three boundary conditions (23)–(25) yield

$$\mathbf{Ax} = 0 \quad (27)$$

where $\mathbf{x} = (e_1, f_1, f_2)^{\mathrm{T}}$ and

$$\mathbf{A} = \begin{bmatrix} -J_0(\lambda_n) & J_0(\lambda_n/\sqrt{k}) & Y_0(\lambda_n/\sqrt{k}) \\ -K^*J_1(\lambda_n) & J_1(\lambda_n/\sqrt{k}) & Y_1(\lambda_n/\sqrt{k}) \\ 0 & J_1(\lambda_n b/\sqrt{k}) & Y_1(\lambda_n b/\sqrt{k}) \end{bmatrix} \quad (28a)$$

for boundary condition (25a) or

$$\mathbf{A} = \begin{bmatrix} -J_0(\lambda_n) & J_0(\lambda_n/\sqrt{k}) & Y_0(\lambda_n/\sqrt{k}) \\ -K^*J_1(\lambda_n) & J_1(\lambda_n/\sqrt{k}) & Y_1(\lambda_n/\sqrt{k}) \\ 0 & J_0(\lambda_n b/\sqrt{k}) & Y_0(\lambda_n b/\sqrt{k}) \end{bmatrix} \quad (28b)$$

for boundary condition (25b), where $K^* = \sqrt{k}/K$. Equations (27) are homogeneous simultaneous equations for e_1, f_1, and f_2. Non-trivial solutions exist if the determinant of the coefficients is zero, i.e.

$$\det \mathbf{A} = 0. \quad (29)$$

Equation (29) can be solved for the eigenvalue λ. For each value of λ, λ_n, which satisfies equation (29), only two of the three equations in (27) are linearly independent, and two of the unknowns, say f_1 and f_2, can be solved from these two equations in terms of the other, say e_1. Thus, the eigenfunction R_n, aside from the constant multiplier e_1, is

$$R_n(r) = \begin{cases} R_{1n} = J_0(\lambda_n r) \\ R_{2n} = A_n J_0(\lambda_n r/\sqrt{k}) + B_n Y_0(\lambda_n r/\sqrt{k}) \end{cases} \quad (30)$$

with

$$A_n = \frac{J_0(\lambda_n)Y_1(\lambda_n/\sqrt{k}) - K^*Y_0(\lambda_n/\sqrt{k})J_1(\lambda_n)}{J_0(\lambda_n/\sqrt{k})Y_1(\lambda_n/\sqrt{k}) - Y_0(\lambda_n/\sqrt{k})J_1(\lambda_n/\sqrt{k})}$$

$$B_n = \frac{K^*J_0(\lambda_n/\sqrt{k})J_1(\lambda_n) - J_0(\lambda_n)J_1(\lambda_n/\sqrt{k})}{J_0(\lambda_n/\sqrt{k})Y_1(\lambda_n/\sqrt{k}) - Y_0(\lambda_n/\sqrt{k})J_1(\lambda_n/\sqrt{k})}.$$

The functions $\{R_n(r)\}$ do not form an orthogonal set, because the first derivative of $R_n(r)$ is discontinuous at $r = 1$ as indicated by equation (19), and therefore the Sturm–Liouville theorem of orthogonality does not apply. The functions, however, can be made orthogonal to each other with respect to a proper choice of weight function which can be found by the theorem of Yeh [16].

Theorem. Given the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + [q(x) + \lambda^2 w(x)]y = 0 \quad (31)$$

where $p(x)$, $q(x)$ and $w(x)$ may be discontinuous at $x = x_1, x_2, \ldots, x_{n-1}$ in the closed interval $x_0 \leqslant x \leqslant x_N$.

If $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the values of this equation satisfying the boundary conditions

$$D_1 y(x_0) - D_2 y'(x_0) = 0 \tag{32}$$

and

$$E_1 y(x_N) - E_2 y'(x_N) = 0 \tag{33}$$

where a prime denotes the derivative and D_1, D_2, E_1, E_2 are arbitrary constants, and these solutions possess the following discontinuities at $x = x_1, x_2, \ldots, x_i, \ldots x_{N-1}$:

$$y(x_i^-) = B_{1i} y(x_i^+) + B_{3i}[b(x)y]'_{x=x_i^+}, \quad i = 1, 2, \ldots, N-1 \tag{34}$$

$$[b(x)y]'_{x=x_i^-} = B_{2i}[b(x)y]'_{x=x_i^+} + B_{4i} y(x_i^+), \quad i = 1, 2, \ldots, N-1 \tag{35}$$

in which B_{1i} and B_{2i} are the constants and $b(x)$ and $b'(x)$ may be discontinuous at $x = x_1, x_2, \ldots, x_{N-1}$, and if $y_1, y_2, y_3, \ldots$ are the solutions corresponding to these values of λ, then the functions $\{y_n(x)\}$ form a system orthogonal with respect to the weight function $w(x)$ over the interval (x_0, x_N) if

$$B_{1i}B_{2i} - B_{3i}B_{4i} = \frac{p(x_i^+)b(x_i^-)}{p(x_i^-)b(x_i^+)}, \quad i = 1, 2, \ldots, N-1. \tag{36}$$

The proof of the theorem can be found in Yeh [16].

To apply the theorem rewrite equation (21) as

$$\frac{d}{dx}\left(F_i r \frac{dR}{dr}\right) + \lambda^2 \frac{F_i}{f(r)} rR = 0, \quad r_{i-1} < r < r_i, \quad i = 1, 2 \tag{37}$$

where $r_0 \equiv 0$ and F_i is a constant within the interval $r_{i-1} < r < r_i$, yet unknown. It is desired to determine the unknown F_i from the above theorem such that the solution, $R_n(r)$, obtained before, be orthogonal with respect to the weight function $F_i r$. That is

$$\sum_{i=1}^{2} \int_{r_{i-1}}^{r_i} R_n(r) R_m(r) F_i r \, dr = 0, \text{ if } n \neq m$$
$$\neq 0, \text{ if } n = m. \tag{38}$$

To determine F_i compare equation (37) with equation (31) and equations (23) and (24) with equations (34) and (35), respectively, to obtain the following relations:

$$b = 1, \quad q = 0, \quad B_{11} = 1, \quad B_{31} = 0,$$
$$B_{21} = K, \quad B_{41} = 0, \quad p = F_i r,$$
$$w = F_i r / f(r) \quad \text{for} \quad r_{i-1} < r < r_i, \quad i = 1, 2. \tag{39}$$

Substituting equations (39) in the condition of orthogonality (36) yields

$$F_2/F_1 = K. \tag{40}$$

Equation (40) implies that the ratio of F_2 to F_1 must be equal to K. The proportional constant is immaterial as it does not affect the orthogonal relation (38). F_i will be taken as the dimensionless quantity

$$F_1 = 1, \quad F_2 = K. \tag{41}$$

Case 1. *Wall temperature boundary condition*

Substitute $\theta = S_2 + \sum_{n=1}^{\infty} R_n(r)\Gamma_n(\tau)$ into equation (15) which yields after rearrangement

$$-\sum_{n=1}^{\infty} R_n(r)\Gamma_n(\tau) + \sum_{n=1}^{\infty} f(r) \frac{1}{r} \frac{d}{dr}\left(r \frac{dR_n(r)}{dr}\right)\Gamma_n(\tau) + g(r) = 0 \tag{42}$$

or upon using equation (21)

$$\sum_{n=1}^{\infty} R_n(r)\Gamma_n(\tau) + \sum_{n=1}^{\infty} \lambda_n^2 R_n(r)\Gamma_n(\tau) - g(r) = 0. \tag{43}$$

Next $g(r)$ is expanded in terms of the eigenfunctions

$$g(r) = \sum_{n=1}^{\infty} G_n R_n(r) \tag{44}$$

where the constants G_n are given by

$$G_n = c \int_0^1 r^3 R_{1n}(r)\, dr \Big/ \left[\int_0^1 R_{1n}(r) r\, dr + \frac{K}{k}\int_1^b R_{2n}^2(r) r \, dr\right], \quad n = 1, 2, \ldots, N. \tag{45}$$

Inserting equation (44) into equation (43) and rearranging gives

$$\sum_{n=1}^{\infty} R_n(r)[\Gamma_n'(\tau) + \lambda_n^2 \Gamma_n(\tau) - G_n] = 0. \tag{46}$$

Since, in general, $R_n(r)$ is not zero, we must have

$$\Gamma_n' + \lambda_n^2 \Gamma_n - G_n = 0, \quad n = 1, 2, \ldots \tag{47}$$

The solution of equation (47) is

$$\Gamma_n = c_n e^{-\lambda_n^2 \tau} + \frac{G_n}{\lambda_n^2}, \quad n = 1, 2, \ldots \tag{48}$$

The temperature distribution is then given by

$$\theta(r, \tau) = S_2 + \sum_{n=1}^{\infty} R_n(r)\left(c_n e^{-\lambda_n^2 \tau} + \frac{G_n}{\lambda_n^2}\right). \tag{49}$$

The coefficients c_n can be obtained by using the initial condition (16) together with the orthogonality properties of the eigenfunctions

$$c_n = -S_2\left(\int_0^1 R_{1n}(r) r\, dr + \frac{K}{k}\int_1^b R_{2n}(r) r\, dr\right) \Big/ \left(\int_0^1 R_{1n}^2(r) r\, dr + \frac{K}{k}\int_1^b R_{2n}^2(r) r\, dr\right) - \frac{G_n}{\lambda_n^2},$$
$$n = 1, 2, \ldots \tag{50}$$

The steady-state temperature distribution can easily be derived by setting the left-hand side of equation (15) to zero and solving for θ, giving

$$\theta(r,\tau) = \begin{cases} S_2 + (c/16)[1 - r^4 + (4/K)\ln b] & 0 \leqslant r \leqslant 1 \\ S_2 + (c/4K)\ln(r/b) & 1 \leqslant r \leqslant b. \end{cases}$$

Also, when the fluid and solid have the same thermophysical properties and viscous dissipation is negligible, equation (49) reduces to

$$\theta(r,\tau) = S_2\left[1 - 2\sum_{n=1}^{\infty} e^{-\lambda_n^2\tau}\frac{J_0(\lambda_n r)}{\lambda_n b J_1(\lambda_n b)}\right]$$

with λ_n, $n = 1,2,\ldots$ being the positive roots of $J_0(\lambda b) = 0$. This is the solution for the temperature distribution in a solid cylinder of infinite length, subjected to a constant surface temperature, see e.g. p. 199 of Carslaw and Jaeger [19].

Case 2. Heat flux boundary condition

First homogenize condition (20a) in the following way. Define

$$\theta(r,\tau) = \psi(r,\tau) + S_1\phi(r). \tag{51}$$

Substitute equation (51) into equations (17) and (20a) to obtain

$$r = 0:\quad \frac{\partial\psi}{\partial r} + S_1\phi'(r) = 0 \tag{52}$$

$$r = b:\quad \frac{\partial\psi}{\partial r} + S_1\phi'(r) = S_1. \tag{53}$$

In order to homogenize equations (52) and (53) we, obviously, must have

$$\phi'(0) = 0 \tag{54}$$

and

$$\phi'(b) = 1. \tag{55}$$

A function $\phi(r)$ which satisfies equations (54) and (55) is, e.g.

$$\phi(r) = \frac{r^2}{2b}. \tag{56}$$

Substituting equation (51) with equation (56) into equation (15) gives

$$\frac{\partial\psi}{\partial\tau} = f(r)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + g^*(r), \quad \tau > 0,\ 0 < r < b \tag{57}$$

where

$$g^*(r) = \begin{cases} cr^2 + 2S_1/b & 0 \leqslant r \leqslant 1 \\ 2S_1 k/b & 1 \leqslant r \leqslant b. \end{cases} \tag{58}$$

Following a similar procedure to that in case 1 yields the following solution:

$$\theta(r,\tau) = G_0^*\tau + \frac{S_1}{b}\left(\frac{r^2}{2} - \frac{1}{4}\right) + \sum_{n=1}^{N} R_n(r)\left(c_n\,e^{-\lambda_n^2\tau} + \frac{G_n^*}{\lambda_n^2}\right) \tag{59}$$

with

$$G_n^* = \left[\int_0^1\left(cr^2 + \frac{2S_1}{b}\right)R_{1n}(r)r\,dr + \frac{K}{k}\int_1^b \frac{2S_1 k}{b}R_{2n}(r)r\,dr\right]\Bigg/\left[\int_0^1 R_{1n}^2(r)r\,dr + \frac{K}{k}\int_1^b R_{2n}^2(r)r\,dr\right], \quad n = 0,1,2,\ldots,N \tag{60}$$

and

$$c_n = -\frac{S_1}{2b}\left[\int_0^1 R_{1n}(r)r^3\,dr + \frac{K}{k}\int_1^b R_{2n}(r)r^3\,dr\right]\Bigg/\left[\int_0^1 R_{1n}^2(r)r\,dr + \frac{K}{k}\int_1^b R_{2n}^2(r)r\,dr\right] - \frac{G_n^*}{\lambda_n^2},$$
$$n = 1,2,\ldots \tag{61}$$

Note that $\lambda_0 = 0$ is an eigenvalue of (28a), which gives $R_{10}(r) = R_{20}(r) = 1$, and yields the linear term in time which appears in equation (59).

When the fluid and solid have identical thermophysical properties and viscous dissipation is negligible, the solution (59) reduces to

$$\theta(r,\tau) = \frac{2S_1}{b}\tau + \frac{S_1}{b}\left[\frac{r^2}{2} - \frac{1}{4} - 2\sum_{n=1}^{\infty} e^{-\lambda_n^2\tau}\frac{J_0(\lambda_n r)}{\lambda_n^2 b^2 J_0(\lambda_n b)}\right]$$

where λ_n, $n = 1,2,\ldots$ are the positive roots of $J_1(\lambda b) = 0$. This is the solution for the temperature distribution in a solid cylinder of infinite length being imposed to an external wall heat flux, see e.g. p. 203 of Carslaw and Jaeger [19].

RESULTS AND DISCUSSION

A new approach was introduced for the solution of conjugated heat transfer problems. The solid and fluid domains are combined and regarded as one domain with discontinuities. A solution by an eigenfunction expansion was presented, the eigenfunctions of which are not orthogonal to each other with respect to the usual weighting function, according to the Sturm–Liouville theorem of orthogonality. These eigenfunctions, however, become orthogonal to each other with respect to a special weighting function, derived from a theorem by Yeh [16]. The main results for the temperature distributions are given by equation (49) for a prescribed wall temperature, and by equation (59) for an imposed wall heat flux.

The variation of the fluid–wall interfacial temperature with time for different values of the conjugation parameter K^* is presented in Tables 1 and 2; the former for a prescribed wall temperature and

Table 1. Temperature variation with time at the solid–fluid interface ($r = 1$), for the parameters $\sqrt{k} = 2$, $b = 1.2$, $c = 1$ and $S_2 = 1$

τ	K^* 0.0005	0.005	0.05	0.5	5
0.001	0.055	0.054	0.052	0.037	0.010
0.002	0.243	0.242	0.232	0.165	0.043
0.003	0.416	0.414	0.397	0.283	0.075
0.004	0.552	0.549	0.528	0.378	0.101
0.005	0.656	0.653	0.628	0.453	0.122
0.006	0.736	0.733	0.706	0.512	0.140
0.007	0.798	0.795	0.766	0.561	0.156
0.008	0.845	0.842	0.812	0.601	0.170
0.009	0.881	0.878	0.848	0.634	0.183
0.01	0.909	0.906	0.876	0.662	0.195
0.02	0.993	0.991	0.972	0.803	0.281
0.03	0.999	0.998	0.985	0.856	0.340
0.04	1.000	0.999	0.988	0.884	0.386
0.05	1.000	0.999	0.990	0.902	0.425
0.06	1.000	0.999	0.992	0.916	0.458
0.07	1.000	0.999	0.993	0.926	0.486
0.08	1.000	0.999	0.994	0.934	0.512
0.09	1.000	0.999	0.994	0.940	0.536
0.10	1.000	0.999	0.995	0.946	0.557
0.20	1.000	1.000	0.998	0.976	0.709
0.30	1.000	1.000	0.999	0.991	0.808
0.40	1.000	1.000	1.000	0.999	0.880
0.50	1.000	1.000	1.001	1.004	0.936
0.60	1.000	1.000	1.001	1.007	0.978
0.70	1.000	1.000	1.001	1.009	1.010
0.80	1.000	1.000	1.001	1.010	1.034
0.90	1.000	1.000	1.001	1.011	1.053
1.00	1.000	1.000	1.001	1.011	1.068
2.00	1.000	1.000	1.001	1.011	1.111
3.00	1.000	1.000	1.001	1.011	1.114

Table 2. Temperature variation with time at the solid–fluid interface ($r = 1$), for the parameters $\sqrt{k} = 2$, $b = 1.2$, $c = 1$ and $S_1 = 1$

τ	K^* 0.0005	0.005	0.05	0.5	5
0.01	0.274	0.269	0.226	0.114	0.100
0.02	0.341	0.336	0.294	0.185	0.159
0.03	0.408	0.403	0.361	0.249	0.207
0.04	0.474	0.469	0.427	0.307	0.249
0.05	0.541	0.536	0.492	0.362	0.287
0.06	0.608	0.602	0.557	0.415	0.322
0.07	0.674	0.669	0.621	0.465	0.356
0.08	0.741	0.735	0.685	0.513	0.388
0.09	0.808	0.801	0.748	0.560	0.418
0.10	0.874	0.867	0.811	0.605	0.448
0.20	1.540	1.529	1.427	1.017	0.712
0.30	2.206	2.188	2.027	1.391	0.955
0.40	2.872	2.846	2.619	1.752	1.192
0.50	3.538	3.503	3.207	2.108	1.428
0.60	4.203	4.160	3.792	2.463	1.664
0.70	4.869	4.817	4.377	2.818	1.900
0.80	5.535	5.474	4.961	3.172	2.135
0.90	6.200	6.131	5.544	3.526	2.371
1.00	6.866	6.787	6.128	3.880	2.607
2.00	13.523	13.354	11.961	7.422	4.963
3.00	20.179	19.921	17.795	10.964	7.319
4.00	26.835	26.487	23.628	14.505	9.676
5.00	33.492	33.054	29.461	18.047	12.032
6.00	40.148	39.621	35.295	21.589	14.388
7.00	46.805	46.187	41.128	25.130	16.745
8.00	53.461	52.754	46.961	28.672	19.101
9.00	60.118	59.321	52.795	32.214	21.457
10.00	66.774	65.887	58.628	35.755	23.814

the latter for an imposed heat flux. The relevant parameters for both tables are $\sqrt{k} = 2$, $b = 1.2$, $c = 1$ and $S_2 = 1$ (Table 1) or $S_1 = 1$ (Table 2). The results are obtained with 50 terms in the series expansion (using more terms did not affect the results to four significant digits).

From Table 1 it may be realized that the temperature decreases with increasing values of the conjugation parameter and that it reaches an expected steady state. Table 2 shows that the temperature decreases with larger values of K^* and also with time. As expected, the temperature does not reach a steady state due to a constant heat flux at the pipe outer wall.

The greater the value of the conjugation parameter $K^* = \sqrt{((\rho c\bar{k})_{\text{fluid}}/(\rho c\bar{k})_{\text{solid}})}$, the greater the effect of the fluid thermophysical properties on the solid–fluid interface temperature. Thus, at a given time, the dimensionless temperature of the solid–fluid interface is closer to the initial system temperature ($\theta = 0$) for greater values of the conjugation parameter.

Table 3 shows a comparison between the results of the present solution and those of Krishan [13], for a case where a constant heat flux is prescribed at the pipe wall. Krishan's [13] solution is not only a short time solution but also a boundary layer type of solution in that the thickness of the thermal layer in the fluid should always be less than R_i. The present eigenfunction solution is not restricted in this fashion. It can be realized that the results are considerably

Table 3. Temperature variation with time at the solid–fluid interface ($r = 1$), for the parameters $\sqrt{k} = 2$, $b = 1.2$, $c = 1$ and $S_1 = 1$: a comparison between the present solution and that of Krishan [13]

	τ	K^* 0.0005	0.005	0.05	0.5	5
Present solution	0.01	0.274	0.269	0.226	0.114	0.100
Krishan [13]		0.204	0.198	0.164	0.128	0.062
Present solution	0.09	0.808	0.801	0.748	0.560	0.418
Krishan [13]		1.954	1.923	1.646	0.841	0.284
Present solution	0.25	1.873	1.858	1.728	1.206	0.835
Krishan [13]		5.204	4.723	4.286	1.624	0.485

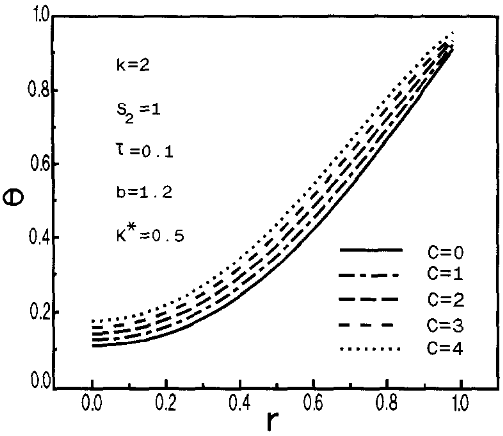

FIG. 1. Temperature variation with distance from the pipe centre for different magnitudes of viscous dissipation.

different from those of Krishan's approximate solution [13], which is supposed to be valid for short time periods only. We do not find a consistent trend in the difference between our solution and that of Krishan. It should be noted that in the case of identical thermophysical properties of the solid and the fluid and negligible viscous dissipation our solutions both for a prescribed wall temperature and for a prescribed wall heat flux, yield the correct solution for heat conduction in a solid cylinder (without the heat generation). In addition, the solution for the case of a given wall temperature yields the correct steady-state solution. These checks lead us to hope that our solutions are correct.

The effect of viscous dissipation on the fluid temperature distribution is depicted in Fig. 1. Here the variation of the temperature with the radius is shown for different magnitudes of viscous dissipation. From this figure, it may be observed that temperatures increase both with the radius and with larger values of viscous dissipation.

There may be situations of opposing heat transfer mechanisms. Heat generation tends to increase the fluid temperature while in the case of heat being extracted at the pipe wall, the latter tends to decrease the fluid temperature. Table 4 shows a case where heat at the pipe wall is being removed. From this table, one can realize that for the higher values of the conjugation parameter heat transfer reversal at the solid–fluid occurs at some stage in the transient. This heat flow reversal occurs sooner for increasing values of the conjugation parameter.

Table 4. Temperature variation with time at the solid–fluid interface ($r = 1$), for the parameters $\sqrt{k} = 2$, $b = 1.2$, $c = 10$ and $S_1 = -1$

τ	K^* 0.0005	0.005	0.05	0.5	5
0.01	−0.274	−0.269	−0.222	−0.083	−0.021
0.02	−0.341	−0.335	−0.284	−0.118	−0.005
0.03	−0.408	−0.401	−0.345	−0.144	0.019
0.04	−0.474	−0.467	−0.404	−0.164	0.046
0.05	−0.541	−0.532	−0.462	−0.179	0.075
0.06	−0.607	−0.598	−0.519	−0.192	0.105
0.07	−0.674	−0.664	−0.576	−0.202	0.135
0.08	−0.740	−0.729	−0.632	−0.210	0.165
0.09	−0.807	−0.795	−0.687	−0.217	0.196
0.10	−0.873	−0.860	−0.742	−0.223	0.226
0.20	−1.539	−1.511	−1.268	−0.241	0.523
0.30	−2.203	−2.159	−1.773	−0.229	0.815
0.40	−2.868	−2.805	−2.266	−0.207	1.107
0.50	−3.532	−3.451	−2.754	−0.181	1.398
0.60	−4.197	−4.096	−3.238	−0.154	1.687
0.70	−4.861	−4.741	−3.721	−0.127	1.981
0.80	−5.526	−5.385	−4.204	−0.099	2.272
0.90	−6.190	−6.030	−4.686	−0.071	2.563
1.00	−6.855	−6.674	−5.167	−0.043	2.854
2.00	−13.499	−13.119	−9.982	0.234	5.766
3.00	−20.143	−19.563	−14.797	0.512	8.678
4.00	−26.787	−26.008	−19.612	0.790	11.590
5.00	−33.431	−32.452	−24.427	1.068	14.502
6.00	−40.075	−38.897	−29.242	1.346	17.413
7.00	−46.719	−45.341	−34.057	1.623	20.325
8.00	−53.363	−51.786	−38.871	1.901	23.237
9.00	−60.007	−58.230	−43.686	2.179	26.149
10.00	−66.651	−64.675	−48.501	2.457	29.061

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TRANSFERT THERMIQUE CONJUGUE VARIABLE POUR UN ECOULEMENT LAMINAIRE DANS UN TUBE

Résumé—Cette étude analyse le transfert thermique conjugué variable dans l'écoulement laminaire pleinement établi hydrodynamiquement et thermiquement dans un tube. Deux cas sont considérés : température pariétale uniforme et densité de flux thermique uniforme à la paroi. Une méthode de séparation des variables traite le fluide et le solide comme une seule région avec certaines discontinuités. Les fonctions propres qui ne sont pas orthogonales entre elles selon la fonction de pondération usuelle en accord avec le théorème de Sturm–Liouville, sont rendues orthogonales à l'aide d'une fonction de pondération spéciale. On conclut que le degré de conjugaison et la dissipation visqueuse peuvent avoir un grand effet sur la distribution de température dans le fluide.

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Zusammenfassung—In der vorliegenden Arbeit wird der zeitlich veränderliche konjugierte Wärmeübergang bei hydrodynamisch und thermisch vollständig entwickelter laminarer Rohrströmung analytisch behandelt. Es werden die Fälle konstanter Wandtemperatur und konstanter Wärmestromdichte in der Rohrwand betrachtet. Eine unkonventionelle Methode der Variablentrennung wird angewandt, welche Fluid und Feststoff als ein Gebiet mit bestimmten Diskontinuitäten behandelt. Die resultierenden Eigenfunktionen sind—bezogen auf die üblichen Gewichtungsfunktionen nach dem Sturm/Liouville-Theorem—nicht orthogonal zueinander. Sie werden jedoch mit Hilfe einer speziellen Gewichtungsfunktion in orthogonalen Zustand gebracht. Es zeigt sich, daß der Grad der Konjugation und der viskosen Dissipation einen starken Einfluß auf die Temperaturverteilung im Fluid haben können.

НЕСТАЦИОНАРНЫЙ СОПРЯЖЕННЫЙ ТЕПЛОПЕРЕНОС ПРИ ЛАМИНАРНОМ ТЕЧЕНИИ В ТРУБЕ

Аннотация—Анализируется нестационарный сопряженный теплоперенос при ламинарном течении в трубе с полностью развитым гидродинамическим и температурным профилями. Исследуются два случая: с постоянной температурой стенки и постоянного теплового потока. Применяется нестандартный метод разделения переменных, в котором жидкость и твердое тело рассматриваются как одна область с некоторыми разрывностями. Полученные в результате собственные функции не являются взаимно ортогональными относительно обычной весовой функции согласно теореме Штурма–Лиувилля, но взаимно ортогональны относительно особой весовой функции. Делается вывод, что степень сопряженности и вязкая диссипация могут оказывать существенное влияние на распределение температур в жидкости.